MATH2040 Linear Algebra II

Tutorial 9

November 10, 2016

1 Examples:

Example 1

Let $V = M_{2\times 2}(\mathbb{R})$ be an inner product space and $T : V \to V$ be a linear operator be defined by $T(A) = A^t$.

(a) Determine whether T is normal, self-adjoint or neither.

(b) Find an orthonormal basis of eigenvectors of T for V, and write down the corresponding eigenvalues.

Solution

(a) Let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the standard ordered basis for $M_{2\times 2}(\mathbb{R})$. Then

$$[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And it is obvious that $[T]_\alpha$ is self-adjoint.

(b) Let $f(t) = \det([T]_\alpha - tI)$ be the characteristic polynomial. Then the eigenvalues are $\lambda_1 = 1$ with multiplicity $m_1 = 3$ and $\lambda_2 = -1$ with multiplicity $m_2 = 1$.

For $\lambda_1 = 1$, $N([T]_\alpha - I) = N\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$

For $\lambda_2 = -1$, $N([T]_\alpha + I) = N\left(\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\right) \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$

So the orthonormal eigenbasis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Example 2

Let T be a linear operator on a complex inner product space V with an adjoint T^*. Prove the following results:

(a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.

(b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = T_0$(zero transformation).

(c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$.

Solution

(a) Note

$$\langle T(x), x\rangle = \langle x, T^*(x)\rangle = \langle x, T(x)\rangle$$

since T is self-adjoint. So

$$\langle T(x), x\rangle = \overline{\langle T(x), x\rangle},$$

therefore, $\langle T(x), x\rangle$ is real for all $x \in V$.

(b) We first substitute $x = x + y \in V$ into the relation $\langle T(x), x\rangle = 0$, i.e.

$$\begin{aligned}
0 &= \langle T(x+y), x+y\rangle \\
&= \langle T(x) + T(y), x+y\rangle \\
&= \langle T(x), x\rangle + \langle T(x), y\rangle + \langle T(y), x\rangle + \langle T(y), y\rangle \\
&= \langle T(x), y\rangle + \langle T(y), x\rangle
\end{aligned}$$

So we have $\langle T(x), y\rangle = -\langle T(y), x\rangle$.

Similarly, we substitute $x = x + iy \in V$ into the same relation, we have

$$\begin{aligned}
0 &= \langle T(x+iy), x+iy\rangle \\
&= \langle T(x) + T(iy), x+iy\rangle \\
&= \langle T(x), x\rangle + \langle T(x), iy\rangle + \langle T(iy), x\rangle + \langle T(iy), iy\rangle \\
&= \langle T(x), iy\rangle + \langle T(iy), x\rangle
\end{aligned}$$

Then, $-i\langle T(x), y\rangle = -i\langle T(y), x\rangle$.

Therefore, $\langle T(x), y\rangle = -\langle T(y), x\rangle = -\langle T(x), y\rangle$ for all $x, y \in V$. We can conclude that $T(x) = 0$ for all $x \in V$, and so $T = T_0$.

(c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then

$$\langle T(x), x\rangle = \overline{\langle T(x), x\rangle} = \overline{\langle x, T^*(x)\rangle} = \langle T^*(x), x\rangle,$$

which is the same as $\langle (T - T^*)(x), x\rangle = 0$ for all $x \in V$. By the result of (b), we have the desired result.

Example 3

An $n \times n$ real matrix A is said to be Gramian matrix if there exists a real matrix B such that $A = B^tB$. Prove that A is Gramian matrix if and only if A is symmetric and all of its eigenvalues are non-negative.

Solution

" $\Rightarrow$ " Suppose A is Gramian matrix, then A is symmetric since $A^t = (B^tB)^t = B^tB = A$.

Let λ be an eigenvalue of A with x be the corresponding normalized eigenvector, then

$$\lambda = \lambda\langle x, x\rangle = \langle Ax, x\rangle = \langle B^tBx, x\rangle = \langle Bx, Bx\rangle \geq 0$$

since for real matrix, $B^t = B^*$.

" $\Leftarrow$ " Suppose A is symmetric and all of its eigenvalues are non-negative. As A is symmetric, then there exists an orthonormal basis $\beta = \{v_1, v_2, \ldots, v_n\}$ such that $[L_A]_\beta$ is diagonal with the i-th diagonal entry is the i-th eigenvalue λ_i.

Let D be a diagonal matrix with i-th diagonal entry to be $\sqrt{\lambda_i}$, which is well-defined because all the eigenvalues are non-negative. Let α be the standard ordereed basis and introduce $[I]_\alpha^\beta$ as the change of coordinate matrix, then A and $[L_A]_\beta$ can be related as the following:

$$A = [I]_\beta^\alpha[L_A]_\beta[I]_\alpha^\beta.$$

Since $D^2 = [L_A]_\beta$, so

$$A = [I]_\beta^\alpha[L_A]_\beta[I]_\alpha^\beta = ([I]_\beta^\alpha D)(D[I]_\alpha^\beta).$$

Note, $([I]_\beta^\alpha)^t[I]_\beta^\alpha = I$ since $\{v_1, v_2, \ldots, v_n\}$ are orthonormal basis, so $([I]_\beta^\alpha)^t = ([I]_\beta^\alpha)^{-1} = [I]_\alpha^\beta$.

In conclusion, we can let $B = D[I]_\alpha^\beta$, then $A = ([I]_\beta^\alpha D)(D[I]_\alpha^\beta) = B^tB$ is a Gramian matrix.

2 Exercises:

Question 1 (Section 6.4 Q2(c):

Let $V = \mathbb{C}^2$ be an inner product space and $T : V \to V$ be a linear operator be defined by $T(a, b) = (2a+ib, a+2b)$.

(a) Determine whether T is normal, self-adjoint or neither.

(b) Find an orthonormal basis of eigenvectors of T for V, and write down the corresponding eigenvalues.

Question 2 (Section 6.4 Q10):

Let T be a self-adjoint operator on a finite-dimensional inner product space V.

(a) Prove that for all $x \in V, ||T(x) + \pm ix||^2 = ||T(x)||^2 + ||x||^2$.

(b) Prove that $T - iI$ is invertible. (that is, to show that $T - iI$ is one-to-one and onto)

(c) Prove that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

Question 3 (Section 6.4 Q12):

Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Hence, prove that T is self-adjoint.

Solution

(Please refer to the practice problem set 9.)